Scientific Annals of Computer Science vol. 36 (1), 2026, pp. 89–106
doi: 10.47743/SACS.2026.1.89

# Second Hyper Zagreb Spectral Radii of Graph Operations

Ahmad BILAL*[1] Muhammad MOBEEN MUNIR[2]

**Abstract**

Chemical graph theory is essential for deriving graph spectral radius, especially in quantum chemistry; a significant link exists between eigenvalues and this invariant of spectral graph theory. This invariant is derived from spectrum, and has numerous applications in various fields and is used to solve many real-life problems. Zagreb-type indices and their spectrum variants are crucial for QSAR/QSPR modeling, chemoinformatics, and computational drug design [16]. The literature contains numerous variants of graph spectral radius. In this article, we concentrate on a novel extension of the spectral invariant called Second Hyper Zagreb spectral radius, which is obtained by replacing the adjacency matrix with the Second Hyper Zagreb matrix. The two graph operations that are primarily covered in this article are splitting and shadow graphs. The connection between the Second Hyper Zagreb spectral radius of these graph operations and Second Hyper Zagreb spectral radius of base graph $G$ is of special relevance to us. By using these operations, we can address the challenge of figuring out the relationship between the spectral radius of the base graph $G$ and the spectral radius of the newly generated graph. The findings of this article improve our understanding and pave the way for future research in spectral graph theory, where graph operations are frequently employed and play an important role in solving real-world problems.

**Keywords:** Second Hyper Zagreb Matrix, Second Hyper Zagreb Eigenvalues, Second Hyper Zagreb Spectral Radius, Splitting Graph, Shadow Graph.

This work is licensed under the Creative Commons Attribution Licence (CC BY)

[1]Department of Physical and Numerical Sciences, University of Rasul, Mandi Bahauddin, Pakistan, Email: ch.ahmadbilal97@gmail.com

[2]Institute of Mathematics, University of the Punjab, Quaid-e-Azam Campus, Lahore, Pakistan.

# 1 Introduction

In this article, we employ only such base graphs that are regular and have no isolated vertex. Graphs are offered as a generic structure for dealing with complex problems encountered in a variety of fields of study, including chemistry, physics, computer science, and data science. Graph theory is particularly essential in today's world since it has a variety of applications in science and engineering, including networking, engine modeling, and atomic bonding. Graph theory has been used to investigate a variety of mathematical structures, including groups, semigroups, rings, and vector spaces. Various intriguing software and theoretical methods for dealing with structure-function information and data mining in this sector have recently been developed. Zagreb indices were initially introduced in [14]. One of the most important spectral graph invariants is graph spectral radius. The concept of graph spectral radius, first introduced in chemical graph theory, has seen significant growth due to its usefulness in graph structure, combinatorics, and matrix theory. Graph spectral radius extends well beyond pure mathematic. It creates tools for studying real-world systems, including biological networks and computer systems. Furthermore, by developing computational approaches, this work has the potential to improve software tools for graph analysis, bridging theoretical and applied research. The techniques utilized here ensure that the conclusions are theoretically sound and practically applicable. In chemical graph theory, the second hyper Zagreb spectral radius, which is obtained by replacing the adjacency matrix with the second hyper Zagreb matrix, is very important and becomes a central point of further and effective research in spectral graph theory. It is particularly interesting in network complexity and molecular branching. Recent trends in research have primarily focused on derivations of bounds, explicit formulas, and the use of various graph operations such as splitting, shadow, comb product, and duplication of graphs. The application of second hyper Zagreb spectral radius in all disciplines of real life has become the focus of modern-world researchers because it is inextricably linked to nanostructure modeling and QSAR/QSPR studies. The authors of [12] defined the second Hyper Zagreb index. In [23], some graph families were considered for indices computation, specifically the second Hyper Zagreb index. The Hyper Zagreb index computation utilizing fuzzy theory is thoroughly discussed in [15] with details applications. Recent research indicates that scientists are particularly interested in demonstrating the crucial role of spectral radius in daily human life. Lenes et al. in [17] specifically focused on the maximum degree matrix and established a strong correlation with the structures of extremal graphs, which has wide-ranging implications for robustness. This serves as a well-known illustration of the importance of spectral radius in resilience and connectivity within

spectral graph theory, clearly showing that spectral radius is not merely a spectral invariant. In [18], scientists specifically construct new bounds for the spectral radii of various families of graphs. These results demonstrate the utility of this invariant for optimization in spectral graph theory and illustrate its influence on the growth of graphs under certain constraints. In [21], the authors used spectral radius to develop model selection frameworks; they make a keen development methodology that uses the particular spectrum to distinguish graph models of different candidates; thus, spectral radius is critically important in the classification and evaluation of graph models. In spectral graph theory one of the other most well known invariant is graph energy. In [20], the second Zagreb energy of splitting and shadow graphs was obtained. In [11], the second Zagreb energy of various graphs was discovered. The upper sharp bound for the multiplicative second hyper Zagreb index was obtained by the authors of [2] using composition, join, corona, and cartesian product. In [1], explicit formulas for second hyper Zagreb index were computed for the complement of different graph operations, including composition, Cartesian product, symmetric difference, strong product, disjunction, tensor product, and join. The authors also computed Second hyper Zagreb index for some more valuable physicochemical structures, including molecular complement graphs of Titania Nanotubes, V-Phenylenic Nanotorus, and V-Phenylenic Nanotube. In [19], the authors computed the second hyper Zagreb index for some well-known graph families, including path graphs, conical graphs, cycle graphs, complete graphs, and complete bipartite graphs. They also computed the second hyper Zagreb index for some well-known graph operations, including symmetric difference, strong product, composition, Cartesian product, disjunction, and tensor product graphs. They also computed the second hyper Zagreb index for nano-toruses and nano-tubes. Gao et established closed form expression for some operation like join and cartesian product of some families of graphs in [13]. Present article also deals with particular kind of graph operations known as splitting and shadow graphs. Shirdel et al discussed hyper Zagreb indices of various graph operations in [22].

Research on graph spectral radius has recently and continues to receive significant attention. Nowadays, researchers are particularly interested in how graph operations affect graph spectral radius. Recent research has examined the energy of the $ISI$ matrix due to the deletion of one edge [6]. Findings of [6] served as impetus and a platform for future research into how even the deletion of a single edge has a significant impact on the spectral parameters, particularly graph energy. Strong connections were found when Bilal et al. expanded their study to investigate the maximum and minimum degree spectral radii [24]. Bilal et al. further contributed during the investigation of $ABC$ spectral radius and $ABC$ energy for various graph

operations [4]. The sensitivity of ISI energy and spectral radii under splitting and shadow graphs was revealed in [9]. Due to the significance of energies in chemical graph theory, Bilal et al. offered a complete analysis and precise formulas for the splitting and shadow graph operations [5]. Most recent research work is developed toward new variants of spectral radius, including SDD spectral radius [7], modified Sombor spectral radius [3],first hyper Zagreb spectral Radius [8], and Zagreb spectral radius [10] under these graph operations, particularly splitting and shadow graphs, and gives an exact formula under these operations. New spectral radius invariants are now widely used in several industries. One such spectral radius is the Second Hyper Zagreb spectral radius, which was created in response to successful applications of this matrix.

Second Hyper Zagreb matrix is denoted by $[HZ_2(G)]$ and have the following entries

$$c_{ij} = \begin{cases} (d_i d_j)^2, & \text{when } u_i \text{ is adjacent to } u_j, \\ 0, & otherwise. \end{cases}$$

Eigenvalues of Second Hyper Zagreb matrix are $\varrho_1, \varrho_2, ..., \varrho_n$.
Second Hyper Zagreb spectral radius $\Re HZ_2(G)$ is defined as

$$\Re HZ_2(G) = \max_{1 \le i \le n} \{|\varrho_i|\}.$$

To create $Spl_q(G)$, add $q$ extra split vertices to each vertex $v_i$ of the original graph $G$. These newly added split vertices $v_i^q$ were then connected to all of $v_i$'s neighbors but not to $v_i$ itself. The vertex set of $Spl_q(G)$ is $V(G) \cup \{ v_i^k : 1 \le i \le n,\ 1 \le k \le q \}$. The edge set of $Spl_q(G)$ becomes $E(G) \cup \{ v_i^k v_j : v_i v_j \in E(G),\ 1 \le k \le m \}$.

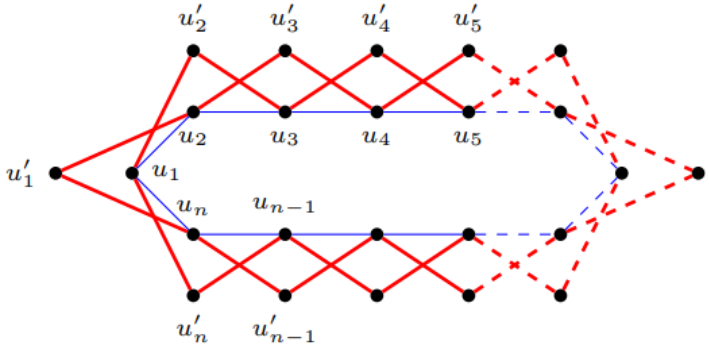


Figure 1: $Spl_1(C_n)$.

To create $Sh_q(G)$, $q$ copies of the original graph $G$ are needed. If vertices are connected in the original graph, they are also connected across $q$ copies. The vertex set of $Sh_q(G)$ is $V(G_1) \cup V(G_2) \cup \cdots \cup V(G_q)$. The edge set of $Sh_q(G)$ becomes $\bigcup_{x=1}^{q} E(G_x) \cup \{\, v_{x,i}v_{s,j} : v_iv_j \in E(G),\ x \neq s \,\}$.

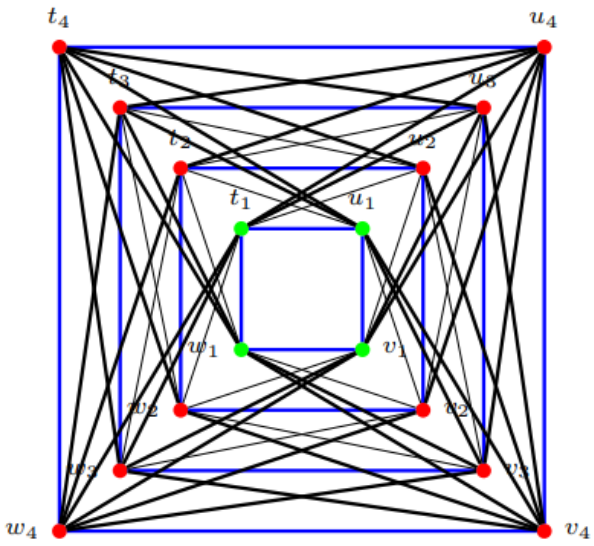


Figure 2: $Sh_4(C_4)$.

## 2 Second Hyper Zagreb Spectral Radii of Generalized Splitting Graphs

This theorem addresses the issue of accurately recovering the Second Hyper Zagreb spectral radius of a generalized splitting graph $\Re HZ_2(Spl_q(G))$ from the original graph $G$. Following result shows that newly obtained invariant is purely a multiplicative scaling, so the generalized splitting operation boosts the invariant by a factor that depends only on splitting parameter $q$.

**Theorem 1.** *For any regular base graph $G$, the Second Hyper Zagreb spectral radius of a generalized splitting graph $\Re HZ_2(Spl_q(G))$ can be correlated with the Second Hyper Zagreb spectral radius of the base graph $G$ as follows*

$$\Re HZ_2(Spl_q(G)) = \frac{(q+1)^4 + (q+1)^2\sqrt{(q+1)^4 + 4q}}{2} \Re HZ_2(G).$$

*Proof.* If $G$ is a regular base graph with degree $z$ and no isolated vertices, the vertex set of $Spl_q(G)$ becomes $\Gamma \cup V(G)$. $\Gamma$ includes additional $q$ vertices for each vertex in $G$. Newly added vertices achieve the degree $z$, but all vertices in the original base graph now have the degree $(q+1)z$. Using the define matrix in 1, if $e$ is an edge connecting original vertices in the base graph, then by definition we have $((q+1)z(q+1)z)^2 = (q+1)^4z^4$. If é is an edge between the original and newly added vertices, we may use the matrix definition to get $((q+1)z*z)^2 = (q+1)^2z^4$. Block matrix representation of $HZ_2\left(Spl_q(G)\right)$ is

$$HZ_2\left(Spl_q(G)\right) =$$

$$\left[\begin{array}{cccc|cccc|c|cccc}
0 & c_{11} & \dots & c_{1q} & 0 & e_{11} & \dots & e_{1q} & \dots & 0 & e_{12} & \dots & e_{1q} \\
c_{21} & 0 & \dots & c_{2q} & e_{21} & 0 & \dots & e_{2q} & \dots & e_{21} & 0 & \dots & e_{2q} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \dots & \vdots & \vdots & \ddots & \vdots \\
c_{q1} & c_{q2} & \dots & 0 & e_{q1} & e_{q2} & \dots & 0 & \dots & e_{q1} & e_{q2} & \dots & 0 \\
\hline
0 & e_{12} & \dots & e_{1q} & 0 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 \\
e_{21} & 0 & \dots & e_{2q} & 0 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \dots & \vdots & \vdots & \ddots & \vdots \\
e_{q1} & e_{q2} & \dots & 0 & 0 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 \\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\hline
0 & e_{12} & \dots & e_{1q} & 0 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 \\
e_{21} & 0 & \dots & e_{2q} & 0 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \dots & \vdots & \vdots & \ddots & \vdots \\
e_{q1} & e_{q2} & \dots & 0 & 0 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0
\end{array}\right]$$

$$HZ_2\left(Spl_q(G)\right) = \begin{bmatrix}
\Bbbk_1 & \Bbbk_2 & \dots & \Bbbk_2 \\
\Bbbk_2 & 0 & \dots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\Bbbk_2 & 0 & \cdots & 0
\end{bmatrix}.$$

$\Bbbk_1 = (q+1)^4 HZ_2(G)$ and $\Bbbk_2 = \left((q+1)^2\right) HZ_2(G)$.

$$HZ_2\left(Spl_q(G)\right) =$$

$$\begin{bmatrix}
\left((q+1)^4\right) HZ_2(G) & \left((q+1)^2\right) HZ_2(G) & \dots & \left((q+1)^2\right) HZ_2(G) \\
\left((q+1)^2\right) HZ_2(G) & 0 & \dots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\left((q+1)^2\right) HZ_2(G) & 0 & \cdots & 0
\end{bmatrix}$$

$$HZ_2\left(Spl_q(G)\right) = HZ_2(G) \bigotimes \begin{bmatrix} (q+1)^4 & (q+1)^2 & \dots & (q+1)^2 \\ (q+1)^2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (q+1)^2 & 0 & \cdots & 0 \end{bmatrix}.$$

Let $[F] = f_{ij}$ having entries

$$f_{ij} = \begin{bmatrix} (q+1)^4 & (q+1)^2 & \dots & (q+1)^2 \\ (q+1)^2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (q+1)^2 & 0 & \cdots & 0 \end{bmatrix}.$$

By leveraging the spectral features of the Kronecker product and the structural limits imposed by low-rank perturbations, we pursue linear algebraic derivation.

$$HZ_2(Spl_q(G)) = HZ_2(G) \otimes F.$$

Where $F$ is the symmetric matrix given below

$$F = \begin{bmatrix} q^4+1+4q^3+6q^2+4q & q^2+2q+1 & \cdots & q^2+2q+1 \\ q^2+2q+1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ q^2+2q+1 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{(q+1)\times(q+1)}.$$

Because $F$ and $HZ_2$ are both real symmetric matrices, their spectra contain real eigenvalues. The spectral theorem also applies here. First, we are interested in finding the set of all eigenvalues of $F$, and it has rank 2 because all rows except the first row are actually the multiples of standard basis vector $e_1$, so all nonzero eigenvalues live in two-dimensional subspace, so $F$ has $q-1$ zero eigenvalues and only two nonzero eigenvalues denoted by $\ell_1$ and $\ell_2$.

$$\mathrm{tr}(F) = \ell_1 + \ell_2 = (q+1)^4.$$

$[F^2] = s_{ij}$ having entries

$$s_{ij} = \begin{bmatrix} (q+1)^8 + q(q+1)^4 & (q+1)^6 & \dots & (q+1)^6 \\ (q+1)^6 & (q+1)^4 & \dots & (q+1)^4 \\ \vdots & \vdots & \ddots & \vdots \\ (q+1)^6 & (q+1)^4 & \cdots & (q+1)^4 \end{bmatrix}.$$

So,

$$\mathrm{tr}(F^2) = \ell_1^2 + \ell_2^2 = (q+1)^8 + 2q(q+1)^4.$$

By utilizing the identity

$$\ell_1^2 + \ell_2^2 = (\ell_1 + \ell_2)^2 - 2\ell_1\ell_2,$$

we are able to compute

$$\ell_1\ell_2 = \frac{(q+1)^8 - \left((q+1)^8 + 2q(q+1)^4\right)}{2} = -q(q+1)^4.$$

Therefore, roots of

$$t^2 - (q+1)^4 t - q(q+1)^4 = 0.$$

are $\ell_1$ and $\ell_2$. after some simplification, we get

$$\ell_1 = \frac{(q+1)^4 + (q+1)^2\sqrt{(q+1)^4 + 4q}}{2},$$
$$\ell_2 = \frac{(q+1)^4 - (q+1)^2\sqrt{(q+1)^4 + 4q}}{2}.$$

The set of all the eigenvalues $HZ_2 \otimes F$ is given by

$$\{\varrho_i\ell_1,\ \varrho_i\ell_2,\ \underbrace{0,\ldots,0}_{q-1 \text{ times}}\}_{i=1}^{n}.$$

It is well known that the spectrum of the Kronecker product $A \otimes B$ consists of all products $\lambda\mu$, where $\lambda$ and $\mu$ range over the eigenvalues of $A$ and $B$, respectively. Hence, the eigenvalues of

$$\Re HZ_2(Spl_q(G)) = \Re HZ_2(G) \otimes F$$

are

$$\left\{\varrho_i\ell_1,\ \varrho_i\ell_2,\ \underbrace{0,\ldots,0}_{q-1 \text{ times}}\right\}_{i=1}^{n}.$$

According to the definition of spectral radius, the maximum absolute eigenvalue of a particular matrix is its spectral radius. Therefore,

$$\Re HZ_2(Spl_q(G)) = \max_{1\le i\le n}\left\{|\varrho_i\ell_1|,\ |\varrho_i\ell_2|\right\}.$$

Due to the fact $|\ell_1| > |\ell_2|$, we have

$$\Re HZ_2(Spl_q(G)) = |\ell_1| \max_{1\leq i\leq n} |\varrho_i| = |\ell_1|\Re HZ_2(G).$$

In the end, we used the explicit expression for $\ell_1$ to get our conclusion

$$\Re HZ_2(Spl_q(G)) = \frac{(q+1)^4 + (q+1)^2\sqrt{(q+1)^4+4q}}{2}\Re HZ_2(G),$$

proof has been completed in this manner. □

The above theorem clearly imparts the following result.

**Proposition 2.1.** $\Re HZ_2(Spl_q(G)) > \Re HZ_2(G)$.

*Proof.* As $q$ increases, the factor $\frac{(q+1)^4+(q+1)^2\sqrt{(q+1)^4+4q}}{2}$ increases strictly, so the result is established pretty easily. However, when $q = 0$, there is equality, which is a trivial fact. Also for large values of $q$, the scaling in the spectrum increases roughly quadratically. □

If a generalized splitting graph is built on the base graph, which is a complete graph on $p$ vertices. Then Proposition 2.2 computes the second hyper Zagreb spectral radius of a generalized splitting graph of $K_p$.

**Proposition 2.2.** *The second hyper Zagreb spectral radius of the q-splitting graph of the complete graph $K_p$, which is $p-1$ regular, is expressed as follows*

$$\Re HZ_2(Spl_q(K_p)) = \frac{(q+1)^4 + (q+1)^2\sqrt{(q+1)^4+4q}}{2}\left((p-1)^5\right).$$

*Proof.* The second hyper Zagreb matrix of $K_p$ takes the following form

$$HZ_2(K_p) = (p-1)^4(J-I),$$

where $I$ is the identity matrix and $J$ is the matrix with all entries one. The eigenvalues of the matrix J are $p, 0, 0, \ldots, 0$. The eigenvalues of $J - I$ are determined by subtracting 1 from the eigenvalues of $J$, yielding $p-1, -1, -1, \ldots, -1$. By multiplying by the scalar factor $(p-1)^4$, we can derive the spectrum $specHZ_2(K_p)$ as follows

$specHZ_2(K_p) =$

$$\begin{pmatrix} (p^4-4p^3+6p^2-4p+1)(p-1) & -p^4+4p^3-6p^2+4p-1 \\ 1 & p-1 \end{pmatrix}.$$

According to the observations, eigenvalue $(p-1)^5$ has multiplicity one, and the eigenvalue $-(p-1)^4$ has multiplicity $p-1$. Using the definition of spectral radius, we arrived at $\Re HZ_2(K_p) = (p-1)^5$. We are able to obtain our final distinction, which is proof of this claim, by inserting $\Re HZ_2(K_p) = (p-1)^5$ in Theorem 1. □

If a generalized splitting graph is built on the base graph, which is a complete bipartite graph on $2p$ vertices. Then Proposition 2.3 computes the second hyper Zagreb spectral radius of a generalized splitting graph of $K_{p,p}$.

**Proposition 2.3.** *The second hyper Zagreb spectral radius of the $q$-splitting graph of the complete bipartite graph $K_{p,p}$, which is $p$ regular, is given by*

$$\Re HZ_2(Spl_q(K_{p,p})) = \frac{(q+1)^4 + (q+1)^2\sqrt{(q+1)^4+4q}}{2}\left(p^5\right).$$

*Proof.*

$$HZ_2\left(K_{p,p}\right) = \left[\begin{array}{cccc|cccc} 0 & 0 & \dots & 0 & p^4 & p^4 & \dots & p^4 \\ 0 & 0 & \dots & 0 & p^4 & p^4 & \dots & p^4 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & p^4 & p^4 & \dots & p^4 \\ \hline p^4 & p^4 & \dots & p^4 & 0 & 0 & \dots & 0 \\ p^4 & p^4 & \dots & p^4 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ p^4 & p^4 & \dots & p^4 & 0 & 0 & \dots & 0 \end{array}\right].$$

The set of all eigenvalues of the above matrix is found to be as follows

$$specHZ_2\left(K_{p,p}\right) = \begin{pmatrix} p^5 & -p^5 & 0 \\ 1 & 1 & 2p-2 \end{pmatrix}.$$

From this spectrum there is clear indication that the eigenvalue $p^5$ has multiplicity one, $-p^5$ has multiplicity one, and 0 has multiplicity $2p-2$. Utilizing the definition of spectral radius, we get $\Re HZ_2(K_{p,p}) = p^5$. To prove this assertion, we can formally put $\Re HZ_2(K_{p,p}) = p^5$ into Theorem 1. □

If a generalized splitting graph is built on the base graph, which is a crown graph on $2p$ vertices. Then Proposition 2.4 computes the second hyper Zagreb spectral radius of a generalized splitting graph of $\Upsilon$.

**Proposition 2.4.** *The second hyper Zagreb spectral radius of the q-splitting graph of the crown graph $\Upsilon$, which is $p-1$ regular, is determined as*

$$\Re HZ_2(Spl_q(\Upsilon)) = \tfrac{(q+1)^4+(q+1)^2\sqrt{(q+1)^4+4q}}{2}\left((p-1)^5\right).$$

*Proof.* $HZ_2(\Upsilon) =$

$$\left[\begin{array}{cccc|cccc} 0 & 0 & \dots & 0 & 0 & (p-1)^4 & \dots & (p-1)^4 \\ 0 & 0 & \dots & 0 & (p-1)^4 & 0 & \dots & (p-1)^4 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & (p-1)^4 & (p-1)^4 & \dots & 0 \\ \hline 0 & (p-1)^4 & \dots & (p-1)^4 & 0 & 0 & \dots & 0 \\ (p-1)^4 & 0 & \dots & (p-1)^4 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ (p-1)^4 & (p-1)^4 & \dots & 0 & 0 & 0 & \dots & 0 \end{array}\right].$$

$$HZ_2(\Upsilon) = \left[\begin{array}{cc} 0 & (p-1)^4(J-I) \\ (p-1)^4(J-I) & 0 \end{array}\right].$$

$$HZ_2(\Upsilon) = (p-1)^4\left[\begin{array}{cc} 0 & (J-I) \\ (J-I) & 0 \end{array}\right].$$

The eigenvalues of $J-I$ are $p-1, -1, -1, \dots, -1$. Consequently, following scalar multiplication by the factor $(p-1)^4$, and block-matrix result the complete set of eigenvalues for the matrix $HZ_2(\Upsilon)$ is determined as follows

$$specHZ_2(\Upsilon) = \left(\begin{array}{cccc} -(p-1)^4 & (p-1)^4 & -(p-1)^5 & (p-1)^5 \\ p-1 & p-1 & 1 & 1 \end{array}\right).$$

The spectrum shows that the eigenvalue $(p-1)^4$ has multiplicity $p-1$, eigenvalue $-(p-1)^4$ has multiplicity $p-1$, eigenvalue $(p-1)^5$ has multiplicity 1, and eigenvalue $-(p-1)^5$ has multiplicity 1. Utilizing the definition of spectral radius, we get $\Re HZ_2(\Upsilon) = (p-1)^5$. To prove this result, put $\Re HZ_2(\Upsilon) = (p-1)^5$ into Theorem 1. □

If a generalized splitting graph is built on the base graph, which is a cycle graph on $p$ vertices. Then Proposition 2.5 computes the second hyper Zagreb spectral radius of a generalized splitting graph of $C_p$.

**Proposition 2.5.** *The second hyper Zagreb spectral radius $\Re HZ_2(Spl_q(C_p))$ of cycle graph $C_p$, which is 2 regular, is founded as*

$$\Re HZ_2(Spl_q(C_p)) = \frac{(q+1)^4 + (q+1)^2\sqrt{(q+1)^4+4q}}{2}\,(32)\,.$$

*Proof.* The matrix $[HZ_2(C_p)]$ is a circulant whose entries are zeros except two nonzero entries in each row and in fact nonzero entries have the same value which is 16 and its due to the adjacency of corresponding vertices. Hence,

$$HZ_2(C_p) = \mathrm{Circ}(0, 16, 0, \ldots, 0, 16).$$

Set of all eigenvalues of above circulant matrix is given below

$$\lambda_t = 32\cos\left(\frac{2\pi t}{p}\right), \quad t = 0, 1, \ldots, p-1.$$

We have the information that

$$-1 \leq \cos\left(\frac{2\pi t}{p}\right) \leq 1 \quad \text{for all values of t,}$$

thus

$$|\lambda_t| = 32\left|\cos\left(\frac{2\pi t}{p}\right)\right| \leq 32.$$

When $t = \frac{p}{2}$ or $t = 0$ the maximum absolute value 32 is obtained which is in fact the spectral radius. We are able to obtain our final distinction, which is proof of this claim, by inserting $\Re HZ_2(C_p) = 32$ in Theorem 1. □

# 3 Second hyper Zagreb spectral radius of Generalized Shadow Graphs

This theorem addresses the issue of accurately recovering the second hyper Zagreb spectral radius of a generalized shadow graph $\Re HZ_2(Sh_q(G))$ from the original graph $G$. However, the result does not involve any transcendental factor. The result actually becomes an algebraic factor of the spectrum of the original graph.

**Theorem 2.** *For any regular base graph G, the Second Hyper Zagreb spectral radius of a generalized shadow graph $\Re HZ_2(Sh_q(G))$ can be correlated with the Second Hyper Zagreb spectral radius of the base graph as follows*

$$\Re HZ_2(Sh_q(G)) = q^5\,\Re HZ_2(G).$$

*Proof.* The second hyper zagreb matrix of the $q$-shadow graph of the base graph $G$ has the notation

$$HZ_2(Sh_q(G)),$$

so

$$HZ_2(Sh_q(G)) = HZ_2(G) \otimes F,$$

rank of the matrix $F \in \mathbb{R}^{q\times q}$ is one

$$F = q^4 \mathbf{1}_q \mathbf{1}_q^\top, \quad \mathbf{1}_q = [1, 1, \ldots, 1]^\top \in \mathbb{R}^q.$$

$F$ has the characteristic polynomial

$$\det(\lambda I_q - F) = \lambda^{q-1}(\lambda - q^5).$$

The set of all the eigenvalues of $F$ is

$$\mathrm{spec}(F) = \{q^5, 0, \ldots, 0\}.$$

The exact eigenvalues of $HZ_2(Sh_q(G))$ are

$$\{\varrho_i\lambda_j \,:\, \varrho_i \in \mathrm{spec}(HZ_2(G)),\ \lambda_j \in \mathrm{spec}(F)\}.$$

Since $F$ contains all zero eigenvalues except one non-zero eigenvalue, we have

$$\mathrm{spec}(HZ_2(Sh_q(G))) = \{q^5\varrho_1, q^5\varrho_2, \ldots, q^5\varrho_n\} \cup \{0\}^{n(q-1)}.$$

The eigenvalues of the matrix $HZ_2(G)$ are $\varrho_1, \ldots, \varrho_n$ .

Due to the definition of spectral radius, the maximum absolute eigenvalue of a particular matrix is its spectral radius. We obtain the second hyper Zagreb spectral radius of the $q$-shadow graph of $G$ in the form

According to the definition of spectral radius, the maximum absolute eigenvalue of a particular matrix is its spectral radius. Therefore,

$$\Re HZ_2(Sh_q(G)) = \max_{1\le i\le n} \left\{|q^5\varrho_1, q^5\varrho_2, \ldots, q^5\varrho_n|,\ |0|\right\}.$$

Due to the fact $|q^5| > |0|$, we have

$$\Re HZ_2(Sh_q(G)) = |q^5| \max_{1\le i\le n} |\varrho_i| = |q^5| \Re HZ_2(G).$$

$$\Re HZ_2(Sh_q(G)) = q^5\, \Re HZ_2(G),$$

So finally, our proof is formally completed. □

The above result immediately computes the spectral radius of the generalized shadow graphs without analyzing the shadowed structure. Since $q^5$ grows strictly with $q$ for $q \geq 1$, the operation amplifies all spectral parameters, including spectral radii, which has deep consequences in molecular physics.

If a $q$-shadow graph is built on the base graph, which is a complete graph on $p$ vertices. Then Proposition 3.1 computes the second hyper Zagreb spectral radius of a generalized shadow graph of $K_p$.

**Proposition 3.1.** *The second hyper Zagreb spectral radius $\Re HZ_2(Sh_q(K_p))$ of $K_p$, which is $p-1$ regular, is given by*

$$\Re HZ_2(Sh_q(K_p)) = (p-1)^5 q^5.$$

*Proof.* We can use our previously acquired result, $\Re HZ_2(K_p) = (p-1)^5$, from Proposition 2.2. By incorporating this result into Theorem 2, we have successfully proved our result. □

If a $q$-shadow graph is built on the base graph, which is a complete bipartite graph on $2p$ vertices. Then Proposition 3.2 computes the second hyper Zagreb spectral radius of a generalized shadow graph of $K_{p,p}$.

**Proposition 3.2.** *The second hyper Zagreb spectral radius of shadow graph of complete bipartite graph, which is p regular, is founded as*

$$\Re HZ_2(Sh_q(K_{p,p})) = p^5 q^5.$$

*Proof.* We can use our previously acquired result, $\Re HZ_2(K_{p,p}) = p^5$, from Proposition 2.3. By incorporating this result into Theorem 2, we have successfully proved our result. □

If a $q$-shadow graph is built on the base graph, which is a crown graph on $2p$ vertices. Then Proposition 3.3 computes the second hyper Zagreb spectral radius of a generalized shadow graph of $\Upsilon$.

**Proposition 3.3.** *The second hyper Zagreb spectral radius $\Re HZ_2(Sh_q(\Upsilon))$ of crown graph, which is $p-1$ regular, is given by*

$$\Re HZ_2(Sh_q(\Upsilon)) = (p-1)^5 q^5.$$

*Proof.* We can use our previously acquired result, $\Re HZ_2(\Upsilon) = (p-1)^5$, from Proposition 2.4. By incorporating this result into Theorem 2, we have successfully proved our result. □

If a $q$-shadow graph is built on the base graph, which is a cycle graph on $p$ vertices. Then Proposition 3.4 computes the second hyper Zagreb spectral radius of a generalized shadow graph of $C_p$.

**Proposition 3.4.** *The second hyper Zagreb spectral radius* $\Re HZ_2(Sh_q(C_p))$ *of* $C_p$*, which is* 2 *regular, is*

$$\Re HZ_2(Sh_q(C_p)) = 32q^5.$$

*Proof.* We can use our previously acquired result, $\Re HZ_2(C_p) = 32$, from Proposition 2.5. By incorporating this result into Theorem 2, we have successfully proved our result. □

# 4 Conclusions

This paper contributes to the burgeoning field of spectral graph theory by investigating spectral radius using a matrix framework, namely the second hyper Zagreb matrix. This research demonstrates that spectral radius is not only a numerical descriptor, it also plays a crucial role in calculating structural changes of graphs under various graph operations, including shadow and splitting graphs. Graph operations significantly influence their spectral behavior, as evidenced by the identified correlations. Obtaining exact results for various graph families sheds light on how graph topology influences the second hyper Zagreb spectral radius, a fundamental property of graph theory. These findings indicate a strong correlation between spectral properties and combinatorial structure, making degree-based spectral invariants more relevant in modern spectral graph theory. This research highlights the potential of the second hyper Zagreb spectral radius as a useful tool for differentiating graph structures under various graph operations. This approach paves the door for future research in spectral graph theory, beyond just a theoretical perspective. This concept can be applied to various graph operations, including comb products, duplication, corona products, and hierarchical graph constructs. The interaction between classical spectral invariants and the second hyper Zagreb spectral radius creates novel extremal problems and characterizations in algebraic graph theory. From an applied perspective, obtained spectrum insights may prove useful in molecular interaction structures, study of large interconnected systems, including neural architectures, communication networks, and optimization models. This work provides a methodology for studying the second hyper Zagreb spectral radius, highlighting its theoretical relevance and potential use in complex network analysis.

## Competing interests

The authors declare that they have no competing interests.

## Data Availability Statement

No data is attached to this article.

## References

[1] Mohammed Alsharafi, Yusuf Zeren, and Abdu Alameri. The second hyper-Zagreb index of complement graphs and its applications of some nano structures. *Asian Journal of Probability and Statistics*, 15(4):54–75, 2021. doi:10.9734/ajpas/2021/v15i430364.

[2] Mathivanan Aruvi, Maria Joseph John, and Ekambaram Ramganesh. The multiplicative second hyper Zagreb index of some graph operations. *Journal of Mathematical and Computational Science*, 10(6):2829–2845, 2020. doi:10.28919/10.28919/jmcs/4997.

[3] Ahmad Bilal and Muhammad Munir. Modified Sombor spectral radii and modified Sombor energies of splitting and shadow graphs. *Scientific Annals of Computer Science*, 34(2):89–112, 2024. doi:10.47743/SACS.2024.2.89.

[4] Ahmad Bilal and Muhammad Mobeen Munir. ABC energies and spectral radii of some graph operations. *Frontiers in Physics*, Volume 10:1053038, 2022. doi:10.3389/fphy.2022.1053038.

[5] Ahmad Bilal and Muhammad Mobeen Munir. Randic and reciprocal Randic spectral radii and energies of some graph operations. *Journal of Intelligent & Fuzzy Systems*, 44(4):5719–5729, 2023. doi:10.3233/JIFS-221938.

[6] Ahmad Bilal and Muhammad Mobeen Munir. ISI energy change due to an edge deletion. *International Journal of Quantum Chemistry*, 124(22):e27501, 2024. doi:10.1002/qua.27501.

[7] Ahmad Bilal and Muhammad Mobeen Munir. SDD spectral radii and SDD energies of graph operations. *Theoretical Computer Science*, 1007:114651, 2024. doi:10.1016/j.tcs.2024.114651.

[8] Ahmad Bilal and Muhammad Mobeen Munir. First hyper Zagreb spectral radii of splitting and shadow graphs. *Scientific Annals of Computer Science*, 35(2):123–138, 2025. doi:10.47743/SACS.2025.2.123.

[9] Ahmad Bilal, Muhammad Mobeen Munir, Muhammad Imran Qureshi, and Muhammad Athar. ISI spectral radii and ISI energies of graph operations. *Frontiers in Physics*, Volume 11 - 2023:1149006, 2023. doi:10.3389/fphy.2023.1149006.

[10] Ahmad Bilal, Muhammad Mobeen Munir, and Yilun Shang. Zagreb spectral radii of splitting and shadow graphs. *Discrete Mathematics, Algorithms and Applications*, page 2550181, 2025. doi:10.1142/S1793830925501812.

[11] Parikshit Das, Sourav Mondal, and Anita Pal. On second Zagreb energy of graphs. *MATCH Communications in Mathematical and in Computer Chemistry*, 92(1):105–131, 2024. doi:10.46793/match.92-1.105D.

[12] Wei Gao, Mohammad Farahani, Muhammad Siddiqui, and Muhammad Kamran Jamil. On the first and second Zagreb and first and second hyper-Zagreb indices of carbon nanocones $CNC_k[n]$. *Journal of Computational and Theoretical Nanoscience*, 13:7475–7482, 01 2017. doi:10.1166/jctn.2016.5742.

[13] Wei Gao, Muhammad Kamran Jamil, and Mohammad Reza Farahani. The hyper-Zagreb index and some graph operations. *Journal of Applied Mathematics and Computing*, 54(1):263–275, 2017. doi:10.1007/s12190-016-1008-9.

[14] Ivan Gutman and Nenad Trinajstić. Graph theory and molecular orbitals. Total $\varphi$-electron energy of alternant hydrocarbons. *Chemical Physics Letters*, 17(4):535–538, 1972. doi:10.1016/0009-2614(72)85099-1.

[15] Sk Rabiul Islam, Bandar Bin Mohsin, and Madhumangal Pal. Hyper-Zagreb index in fuzzy environment and its application. *Heliyon*, 10(16):e36110, 2024. doi:10.1016/j.heliyon.2024.e36110.

[16] Ovidiu Ivanciuc. Chemical graphs, molecular matrices and topological indices in chemoinformatics and quantitative structure-activity relationships. *Current Computer-Aided Drug Design*, 9(2):153–163, 2013. doi:10.2174/1573409911309020002.

[17] Eber Lenes, Luis Medina, María Robbiano, and Jonnathan Rodríguez. On the spectral radius of the maximum degree matrix of graphs. *Mathematics*, 13(11), 2025. doi:10.3390/math13111769.

[18] Chunmeng Liu, Jiang Zhou, and Changjiang Bu. The high order spectral extremal results for graphs and their applications. *Discrete Applied Mathematics*, 357:209–214, 2024. doi:10.1016/j.dam.2024.06.017.

[19] Abdelhafid Modabish, Abdu Alameri, and Mohammed Alsharafi. The second hyper-Zagreb index of graph operations. *Journal of Mathematical and Computational Science*, 11(2):1455–1469, 2021. doi:10.28919/jmcs/5341.

[20] Mitesh J. Patel, Kajal S. Baldaniya, and Ashika Panicker. More on second Zagreb energy of graphs. *Open Journal of Discrete Applied Mathematics*, 6(2):7–13, 2023. doi:10.30538/psrp-odam2023.0084.

[21] Jairo Iván Peña Hidalgo and Jonathan R. Stewart. Model selection for network data based on spectral information. *Applied Network Science*, 9(1):36, 2024. doi:10.1007/s41109-024-00640-4.

[22] Gholam Hassan Shirdel, Hassan Rezapour, and A. M. Sayadi. The hyper-Zagreb index of graph operations. *Iranian Journal of Mathematical Chemistry*, 4(2):213–220, 2013. doi:10.22052/ijmc.2013.5294.

[23] Girish V. Rajasekharaiah and Usha P. Murthy. Hyper-Zagreb indices of graphs and its applications. *Journal of Algebra Combinatorics Discrete Structures and Applications*, 8(1):9–22, 2021.

[24] Xiujun Zhang, Ahmad Bilal, M. Mobeen Munir, and Hafiz Mutte ur Rehman. Maximum degree and minimum degree spectral radii of some graph operations. *Mathematical Biosciences and Engineering*, 19(10):10108–10121, 2022. doi:10.3934/mbe.2022473.

**© Scientific Annals of Computer Science 2026**